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Stationary distribution of a stochastic reaction–diffusion predator–prey model with additional food, fear effect and anti-predator behavior☆

Haokun Qi, Jiani Jin, Bing Liu *, Baolin Kang

School of Mathematics, Anshan Normal University, Anshan Liaoning 114007, PR China

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ABSTRACT

The stationary distribution, as a fundamental concept in stochastic processes, is of great significance for exploring the long-term behavior and stability of populations. In this paper, a stochastic reaction–diffusion predator–prey model with additional food, fear effect and anti-predator behavior is proposed, in which the stochastic fluctuations are characterized by a Ornstein–Uhlenbeck process. We proved the existence and uniqueness of the stationary distribution of the stochastic model by constructing the Lyapunov function. Moreover, this study extends the work of Qi and Liu (2024).

1. Introduction

The study of long-term behavior and stability of populations is of great significance for revealing ecological laws, protecting the ecological environment, promoting sustainable development, and advancing research and applications in related fields. The long-term behavior and stability of populations are not only directly influenced by interspecies factors, but also influenced by many indirect factors. Factors such as fear effect, additional food, and anti-predator behavior have been hot topics of particular concern for ecologists in recent years. The fear effect can affect the birth rate, foraging behavior, and even the distribution of the entire prey population [1,2]. Additional food may make predators live better and hunt more skillfully [3,4]. Anti-predator behavior undoubtedly increases the difficulty of predation, making the dynamic relationship between predators and prey more complex [5,6]. Therefore, based on [7], we propose the following stochastic reaction–diffusion predator–prey model with additional food, fear effect and

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* Corresponding author.
E-mail address: liubing@asnc.edu.cn (B. Liu).

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anti-predator behavior

$$
\begin{cases}
\mathrm{d}u = \left[d_u \Delta u + \dfrac{ru}{1+kv} - \xi_m u - \alpha_1 u^2 - \dfrac{\beta uv}{\epsilon + \delta\gamma + u + \rho v}\right]\mathrm{d}t + \kappa_1 u \mathrm{d}W_1(t), \text{ in } \mathbb{R}^+ \times \mathcal{O}, \\
\mathrm{d}v = \left[d_v \Delta v + \dfrac{c\beta(u+\gamma)v}{\epsilon + \delta\gamma + u + \rho v} - \zeta_m v - \alpha_2 v^2 - \eta uv\right]\mathrm{d}t + \kappa_2 v \mathrm{d}W_2(t), \text{ in } \mathbb{R}^+ \times \mathcal{O}, \\
\dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0, \text{ on } \mathbb{R}^+ \times \partial\mathcal{O}, \\
u(0,x) = u_0(x) \geq 0, \ v(0,x) = v_0(x) \geq 0, \text{ in } \mathcal{O},
\end{cases}
\tag{1}
$$

where u and v represent the biomass of prey and predator, respectively. r and ξ_m stand for the birth rate and mortality rate of prey. α_1 and α_2 stand for the coefficient of intra-specific interference among preys and predators. k is the fear level induced by predator. β is the rate of predation. c and ζ_m respectively are the conversion efficiency and mortality rate of predator. γ is the quantity of the additional food supplied to the predator, and ρ is the strength of mutual interference among predators. ϵ is normalization coefficient relating the densities of prey and predator to the environment in which they interact. δ is the ratio between the handling times towards the additional food and the prey. η is the rate of anti-predator behavior of prey to the predator. $d_u \geq 0$ and $d_v \geq 0$ represent the diffusion rate of prey and predator, respectively. Δ is the usual Laplacian operator in $\Omega \subset \mathbb{R}$, $\mathcal{O}$ is a bounded smooth domain of $\mathbb{R}^n (n \geq 1)$, $\mathbf{n}$ is the outward normal to $\partial\mathcal{O}$. $\kappa_i (i = 1,2)$ represent the intensities of the white noise. $W(t) = (W_1(t), W_2(t))$ is a mutually independent Brownian motion denoted on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions.

However, the traditional stochastic models often use white noise to simulate this randomness, but the white noise model may experience strong noise fluctuations when the time approaches zero, which is inconsistent with the actual situation. In order to better describe and predict the long-term behavior and stability of populations under the influence of random noise, the Ornstein–Uhlenbeck process is introduced into the stochastic population model [8–10]. The Ornstein–Uhlenbeck process is a mean regression process that describes the random behavior of a variable fluctuating around its mean and has the characteristic of regressing to the mean as time approaches infinity. This characteristic enables the Ornstein–Uhlenbeck process to more accurately simulate the random fluctuations in population size in actual ecosystems. Therefore, if we directly apply the Ornstein–Uhlenbeck process to disturb the mortality rate of ξ_m and ζ_m, we can obtain the model

$$
\begin{cases}
\mathrm{d}\xi = [\theta_1(\xi_m - \xi(t))]\mathrm{d}t + \varpi_1 \mathrm{d}W_1(t), \\
\mathrm{d}\zeta = [\theta_2(\zeta_m - \zeta(t))]\mathrm{d}t + \varpi_2 \mathrm{d}W_2(t),
\end{cases}
\tag{2}
$$

where $\theta_1 \geq 0$ and $\theta_2 \geq 0$ are the speeds of reversion, $\varpi_1 \geq 0$ and $\varpi_2 \geq 0$ are the intensities of volatility. $\xi_m \geq 0$ and $\zeta_m \geq 0$ are the long-term average levels used to measure natural mortalities $\xi(t)$ and $\zeta(t)$, respectively.

Combined with model (1) and (2), we propose the following stochastic reaction–diffusion predator–prey model with additional food, fear effect and anti-predator behavior, in which the stochastic fluctuations are characterized by a Ornstein–Uhlenbeck process

$$
\begin{cases}
\mathrm{d}u = \left[d_u \Delta u + \dfrac{ru}{1+kv} - \xi_m u - \alpha_1 u^2 - \dfrac{\beta uv}{\epsilon + \delta\gamma + u + \rho v} - (\xi_0 - \xi_m)\mathrm{e}^{-\theta_1 t} u\right]\mathrm{d}t + \sigma_1(t) u \mathrm{d}W_1(t), \text{ in } \mathbb{R}^+ \times \mathcal{O}, \\
\mathrm{d}v = \left[d_v \Delta v + \dfrac{c\beta(u+\gamma)v}{\epsilon + \delta\gamma + u + \rho v} - \zeta_m v - \alpha_2 v^2 - \eta uv - (\zeta_0 - \zeta_m)\mathrm{e}^{-\theta_2 t} v\right]\mathrm{d}t + \sigma_2(t) v \mathrm{d}W_2(t), \text{ in } \mathbb{R}^+ \times \mathcal{O}, \\
\dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0, \text{ on } \mathbb{R}^+ \times \partial\mathcal{O}, \\
u(0,x) = u_0(x) \geq 0, \ v(0,x) = v_0(x) \geq 0, \text{ in } \mathcal{O},
\end{cases}
\tag{3}
$$

where $\xi_0 = \xi(0)$, $\zeta_0 = \zeta(0)$, $\sigma_1(t) = \frac{\varpi_1}{\sqrt{2\theta_1}}\sqrt{1 - \mathrm{e}^{-2\theta_1 t}}$, $\sigma_2(t) = \frac{\varpi_2}{\sqrt{2\theta_2}}\sqrt{1 - \mathrm{e}^{-2\theta_2 t}}$. Let $\check{\sigma}_1 = \frac{\varpi_1}{\sqrt{2\theta_1}}$, $\check{\sigma}_2 = \frac{\varpi_2}{\sqrt{2\theta_2}}$.

For the stochastic model (3), the stationary distribution is of great significance for exploring the long-term behavior and stability of populations [11]. There are few studies on the stationary distribution of the stochastic reaction–diffusion models [7,8,12]. Thus, we apply the strategy in [13] to investigate the existence and uniqueness of stationary distribution, which is presented in Section 2.

2. Stationary distribution

According to Ref. [13], we define $\mathcal{H} := \left\{\phi : \phi \in L^2(\mathcal{O}), \ \frac{\partial \phi}{\partial x_i} \in L^2(\mathcal{O}), \ i = 1,2\right\}$, and $\mathcal{H}^{-1}$ is the dual space of $\mathcal{H}$. Denote $\|\cdot\|$ is the norms in $\mathcal{H}$, and $\langle \cdot, \cdot \rangle$ is the duality product between $\mathcal{H}$ and $\mathcal{H}^{-1}$. Denote $\mathfrak{L}(\mathbb{H})$ to be the set of all bounded continuous real-valued functions over $\mathbb{H}$, where $\mathbb{H} = \mathcal{H} \times \mathcal{H}$. Let $\mathcal{B}(\mathbb{H})$ is the Borel algebra over $\mathbb{H}$, then $\mathcal{P}(\mathbb{H})$ is the space of all probability measures over $(\mathcal{O}, \mathcal{B}(\mathbb{H}))$.

Definition 2.1 (*[14]*)**.** For all solutions $w(t,x) = (u(t,x), v(t,x)) \in \mathbb{H}$ of model (3), if exists a probability measure $\pi \in \mathcal{P}(\mathbb{H})$ such that

$$
\pi(g) = \pi\left(\mathbb{P}_t g\right),
$$

where

$$
\pi(g) := \int_{\mathbb{H}} g(\phi)\pi(\mathrm{d}\phi) \text{ and } \mathbb{P}_t g(\phi) := \mathbb{E}g(w(t,x,\phi)), \ g \in \mathfrak{L}(\mathbb{H}).
$$

then a probability measure $\pi \in \mathcal{P}(\mathbb{H})$ is a stationary distribution.

Moreover, for $\pi_1, \pi_2 \in \mathcal{P}(\mathbb{H})$, define a metric on $\mathcal{P}(\mathbb{H})$ by

$$d(\pi_1, \pi_2) = \sup_{g \in \mathcal{M}} \left| \int_{\mathbb{H}} g(\phi)\pi_1(\mathrm{d}\phi) - \int_{\mathbb{H}} g(\varphi)\pi_2(\mathrm{d}\varphi) \right|,$$

where

$$\mathcal{M} := \{g : \mathbb{H} \to \mathbb{R},\ |g(\phi) - g(\varphi)| \le \|\phi - \varphi\|,\ \forall\ \phi, \varphi \in \mathbb{H} \text{ and } |g(\cdot)| \le 1\}.$$

Hence, $\mathcal{P}(\mathbb{H})$ is complete under the metric $d(\cdot,\cdot)$.

Lemma 2.1. *There exists a positive constant K such that*

$$\int_{\mathcal{O}} \left(u(t,x) + \frac{1}{c} v(t,x) \right) \mathrm{d}x \le K \text{ a.s.}$$

Proof. Define function $V(t,x) = \int_{\mathcal{O}} \left(u(t,x) + \frac{1}{c} v(t,x) \right) \mathrm{d}x$. Applying Itô's formula yields

$$\begin{aligned}
\mathrm{d}V(t,x) = & \int_{\mathcal{O}} \Bigg[\frac{ru(t,x)}{1+kv(t,x)} - \xi_m u(t,x) - \alpha_1 u^2(t,x) - (\xi_0 - \xi_m)\mathrm{e}^{-\theta_1 t} u(t,x) + \frac{\beta\gamma v(t,x)}{\epsilon + \delta\gamma + u(t,x) + \rho v(t,x)} \\
& - \frac{\zeta_m}{c} v(t,x) - \frac{\alpha_2}{c} v^2(t,x) - \frac{\eta}{c} u(t,x)v(t,x) - \frac{1}{c}(\zeta_0 - \zeta_m)\mathrm{e}^{-\theta_2 t} v(t,x) + d_u \Delta u(t,x) + \frac{d_v}{c} \Delta v(t,x) \Bigg] \mathrm{d}x\mathrm{d}t \\
& + \int_{\mathcal{O}} \sigma_1(t)u(t,x)\mathrm{d}x\mathrm{d}W_1(t) + \int_{\mathcal{O}} \frac{\sigma_2(t)}{c} v(t,x)\mathrm{d}x\mathrm{d}W_2(t) \\
\le & \left(\frac{(r + \zeta_m - \xi_m)^2}{4\alpha_1} + \frac{c\beta^2\gamma^2}{4\zeta_m(\epsilon + \delta\gamma)^2} \right) \mathrm{d}t - \zeta_m \int_{\mathcal{O}} \left(u(t,x) + \frac{1}{c} v(t,x) \right) \mathrm{d}x\mathrm{d}t \\
& + \int_{\mathcal{O}} \sigma_1(t)u(t,x)\mathrm{d}x\mathrm{d}W_1(t) + \int_{\mathcal{O}} \frac{\sigma_2(t)}{c} v(t,x)\mathrm{d}x\mathrm{d}W_2(t) \\
\le & \left[K_1 - \zeta_m V(t,x) \right] \mathrm{d}t + \int_{\mathcal{O}} \sigma_1(t)u(t,x)\mathrm{d}x\mathrm{d}W_1(t) + \int_{\mathcal{O}} \frac{\sigma_2(t)}{c} v(t,x)\mathrm{d}x\mathrm{d}W_2(t),
\end{aligned}$$

where $K_1 = \frac{(r+\zeta_m-\xi_m)^2}{4\alpha_1} + \frac{c\beta^2\gamma^2}{4\zeta_m(\epsilon+\delta\gamma)^2}$.

Now, the standard method in Lemma 2.3 in [15] can prove this Lemma, so it will not be repeated. □

Lemma 2.2 ([13]). *Suppose that for any bounded subset $\mathbb{U} \subset \mathbb{H}$ and $\ell \ge 1$, we have*

(i) $\sup_{t \ge 0} \sup_{\phi \in \mathbb{U}} \mathbb{E}\|w(t,x,\phi)\|^{\ell} < \infty$;

(ii) $\lim_{t \to \infty} \sup_{\phi,\varphi \in \mathbb{U}} \mathbb{E}\|w(t,x,\phi) - w(t,x,\varphi)\|^{\ell} = 0$,

then the process $w(t,x,\phi)$ has a stationary distribution.

To prove the existence and uniqueness of the stationary distribution of model (3), we only need to verify that (i) and (ii) of Lemma 2.2 are satisfied, that is, to prove the following Theorems 2.1 and 2.2.

Theorem 2.1. *For any $\ell \ge 1$, we have*

$$\sup_{0 \le t \le T} \mathbb{E} \left(\|u(t,x)\|^{\ell} + \|v(t,x)\|^{\ell} \right) \le C_2,$$

where $C_2 = 2\left(\|u_0\|^{\ell} + \|v_0\|^{\ell} \right) \mathrm{e}^{2C_1 T}$ is a constant, where

$$C_1 = \max \left\{ \ell r + \frac{1}{2}\ell(\ell-1)\check{\sigma}_1^2 + 8\sqrt{2}\ell^2\check{\sigma}_1^2, \ell c\beta \left(\frac{\gamma}{\epsilon + \delta\gamma} + 1 \right) + \frac{1}{2}\ell(\ell-1)\check{\sigma}_2^2 + 8\sqrt{2}\ell^2\check{\sigma}_2^2 \right\}.$$

Proof. First, for $\ell \ge 1$, applying Itô's Lemma to $\|u(t,x)\|^{\ell} + \|v(t,x)\|^{\ell}$ yields

$$\begin{aligned}
& \|u(t,x)\|^{\ell} + \|v(t,x)\|^{\ell} \\
= & \|u(0,x)\|^{\ell} + \|v(0,x)\|^{\ell} + \int_0^t \Bigg(\ell \|u(r,x)\|^{\ell-2} \left\langle u(r,x), \frac{ru(r,x)}{1+kv(r,x)} \right\rangle - \ell\xi_m \|u(r,x)\|^{\ell} - \ell\alpha_1 \|u(r,x)\|^{1+\ell} \\
& - \ell \|u(r,x)\|^{\ell-2} \left\langle u(r,x), \frac{\beta u(r,x)v(t,x)}{\epsilon + \delta\gamma + u(r,x) + \rho v(r,x)} \right\rangle - \ell(\xi_0 - \xi_m)\mathrm{e}^{-\theta_1 t}\|u(r,x)\|^{\ell}
\end{aligned}$$

$$
\begin{aligned}
&+\ell\|v(r,x)\|^{\ell-2}\left\langle v(r,x),\frac{c\beta(u(r,x)+\gamma)v(r,x)}{\epsilon+\delta\gamma+u(r,x)+\rho v(r,x)}\right\rangle-\ell\zeta_m\|v(r,x)\|^{\ell}-\ell\alpha_2\|v(r,x)\|^{1+\ell}\\
&-\ell\eta\|v(r,x)\|^{\ell-2}\langle v(r,x),u(r,x)v(r,x)\rangle-\ell(\zeta_0-\zeta_m)\mathrm{e}^{-\theta_2 t}\|v(r,x)\|^{\ell}+\ell d_u\|u(r,x)\|^{\ell-2}\langle\Delta u(r,x),u(r,x)\rangle\\
&+\ell d_v\|v(r,x)\|^{\ell-2}\langle\Delta v(r,x),v(r,x)\rangle+\frac{1}{2}\ell(\ell-1)\left(\sigma_1^2(r)\|u(r,x)\|^{\ell}+\sigma_2^2(r)\|v(r,x)\|^{\ell}\right)\Bigg)\mathrm{d}r\\
&+\int_0^t\ell\sigma_1(r)\|u(r,x)\|^{\ell}\mathrm{d}W_1(r)+\int_0^t\ell\sigma_2(r)\|v(r,x)\|^{\ell}\mathrm{d}W_2(r).
\end{aligned}
\tag{4}
$$

Taking the expectation and $\sup(\cdot)$ to (4), and then using Young inequality and Burkholder–Davis–Gundy inequality [16], one has

$$
\begin{aligned}
&\sup_{0\le t\le T}\mathbb{E}(\|u(t,x)\|^{\ell}+\|v(t,x)\|^{\ell})\\
\le&\mathbb{E}(\|u(0,x)\|^{\ell}+\|v(0,x)\|^{\ell})+\sup_{0\le t\le T}\mathbb{E}\int_0^t\Bigg(\ell r\|u(r,x)\|^{\ell}+\ell c\beta\left(\frac{\gamma}{\epsilon+\delta\gamma}+1\right)\|v(r,x)\|^{\ell}\\
&-\ell d_u\|u(r,x)\|^{\ell-2}\|\nabla u(r,x)\|^2-\ell d_v\|v(r,x)\|^{\ell-2}\|\nabla v(r,x)\|^2+\frac{1}{2}\ell(\ell-1)\left(\sigma_1^2(r)\|u(r,x)\|^{\ell}+\sigma_2^2(r)\|v(r,x)\|^{\ell}\right)\Bigg)\mathrm{d}r\\
&+\sup_{0\le t\le T}\mathbb{E}\left|\int_0^t\ell\sigma_1(r)\|u(r,x)\|^{\ell}\mathrm{d}W_1(r)\right|+\sup_{0\le t\le T}\mathbb{E}\left|\int_0^t\ell\sigma_2(r)\|v(r,x)\|^{\ell}\mathrm{d}W_2(r)\right|\\
\le&\mathbb{E}\left(\|u(0,x)\|^{\ell}+\|v(0,x)\|^{\ell}\right)+\sup_{0\le t\le T}\mathbb{E}\int_0^t\left(\ell r+\frac{1}{2}\ell(\ell-1)\check{\sigma}_1^2\right)\|u(r,x)\|^{\ell}\mathrm{d}r\\
&+\sup_{0\le t\le T}\mathbb{E}\int_0^t\left(\ell c\beta\left(\frac{\gamma}{\epsilon+\delta\gamma}+1\right)+\frac{1}{2}\ell(\ell-1)\check{\sigma}_2^2\right)\|v(r,x)\|^{\ell}\mathrm{d}r\\
&+4\sqrt{2}\sup_{0\le t\le T}\mathbb{E}\|u(t,x)\|^{\frac{\ell}{2}}\left[\int_0^t\ell^2\check{\sigma}_1^2\|u(r,x)\|^{\ell}\mathrm{d}r\right]^{\frac{1}{2}}+4\sqrt{2}\sup_{0\le t\le T}\mathbb{E}\|v(t,x)\|^{\frac{\ell}{2}}\left[\int_0^t\ell^2\check{\sigma}_2^2\|v(r,x)\|^{\ell}\mathrm{d}r\right]^{\frac{1}{2}}\\
\le&\mathbb{E}\left(\|u(0,x)\|^{\ell}+\|v(0,x)\|^{\ell}\right)+\sup_{0\le t\le T}\mathbb{E}\int_0^t\left(\ell r+\frac{1}{2}\ell(\ell-1)\check{\sigma}_1^2\right)\|u(r,x)\|^{\ell}\mathrm{d}r+\frac{1}{2}\sup_{0\le t\le T}\mathbb{E}\left(\|u(t,x)\|^{\ell}+\|v(t,x)\|^{\ell}\right)\\
&+\sup_{0\le t\le T}\mathbb{E}\int_0^t\left(\ell c\beta\left(\frac{\gamma}{\epsilon+\delta\gamma}+1\right)+\frac{1}{2}\ell(\ell-1)\check{\sigma}_2^2\right)\|v(r,x)\|^{\ell}\mathrm{d}r+8\sqrt{2}\sup_{0\le t\le T}\mathbb{E}\int_0^t\ell^2\left(\check{\sigma}_1^2\|u(r,x)\|^{\ell}+\check{\sigma}_2^2\|v(r,x)\|^{\ell}\right)\mathrm{d}r\\
\le&\mathbb{E}\left(\|u(0,x)\|^{\ell}+\|v(0,x)\|^{\ell}\right)+C_2\sup_{0\le t\le T}\mathbb{E}\int_0^t\left(\|u(r,x)\|^{\ell}+\|v(r,x)\|^{\ell}\right)\mathrm{d}r+\frac{1}{2}\sup_{0\le t\le T}\mathbb{E}\left(\|u(t,x)\|^{\ell}+\|v(t,x)\|^{\ell}\right).
\end{aligned}
$$

Hence

$$
\sup_{0\le t\le T}\mathbb{E}\left(\|u(t,x)\|^{\ell}+\|v(t,x)\|^{\ell}\right)\le 2\mathbb{E}\left(\|u_0\|^{\ell}+\|v_0\|^{\ell}\right)+2C_2\sup_{0\le t\le T}\mathbb{E}\int_0^t\left(\|u(r,x)\|^{\ell}+\|v(r,x)\|^{\ell}\right)\mathrm{d}r.
$$

By applying the Gronwall's inequality, one gets

$$
\sup_{0\le t\le T}\mathbb{E}\left(\|u(t,x)\|^{\ell}+\|v(t,x)\|^{\ell}\right)\le 2\left(\|u_0\|^{\ell}+\|v_0\|^{\ell}\right)\mathrm{e}^{2C_1T}:=C_2.
$$

The proof is completed. □

Theorem 2.2. *Suppose that for any constant $\ell\ge 1$ such that* $\max\{H_1,H_2\}<0$, *where*

$$
\begin{aligned}
H_1=&\ell r+(\ell rk-rk+\eta+2\ell\alpha_1)K+\frac{\ell\beta(1+c+\rho)}{\rho}+\frac{c\beta\gamma}{\rho(\epsilon+\delta\gamma)}-\beta-\ell\mathfrak{I}d_u-\ell\hat{\xi}+\frac{1}{2}\ell(\ell-1)\check{\sigma}_1^2,\\
H_2=&(rk+\eta+2\ell\alpha_2)K+\beta+\frac{c\beta\gamma[\ell(1+\rho)-1]}{\rho(\epsilon+\delta\gamma)}-\ell\mathfrak{I}d_v-\ell\hat{\zeta}+\frac{1}{2}\ell(\ell-1)\check{\sigma}_2^2,
\end{aligned}
$$

with

$$
\mathfrak{I}=\min\left\{\inf_{\imath\in\mathbb{H}}\frac{\|\nabla\imath(t,x,\phi,\varphi)\|^2}{\|\imath(t,x,\phi,\varphi)\|^2},\inf_{\jmath\in\mathbb{H}}\frac{\|\nabla\jmath(t,x,\phi,\varphi)\|^2}{\|\jmath(t,x,\phi,\varphi)\|^2}\right\},\ \hat{\xi}=\min\{\xi_m,\xi_0\},\ \hat{\zeta}=\min\{\zeta_m,\zeta_0\}.
$$

Then the model (3) has a unique stationary distribution $\pi\in\mathcal{P}(\mathbb{H})$.

Proof. For all $\ell\ge 1$, $\lambda>0$, denote function

$$
\begin{aligned}
\Xi(t,x,\phi,\varphi)=&\mathrm{e}^{\lambda t}\left(\|u(t,x,\phi)-u(t,x,\varphi)\|^{\ell}+\|v(t,x,\phi)-v(t,x,\varphi)\|^{\ell}\right)\\
=&\mathrm{e}^{\lambda t}\left(\|\imath(t,x,\phi,\varphi)\|^{\ell}+\|\jmath(t,x,\phi,\varphi)\|^{\ell}\right).
\end{aligned}
$$

By using Itô's Lemma yields

$$
\begin{aligned}
\mathrm{d}\Xi(t,x,\phi,\varphi) =& \lambda\Xi(t,x,\phi,\varphi)\mathrm{d}t + \mathrm{e}^{\lambda t}\Bigg[\ell\|\imath(t,x,\phi,\varphi)\|^{\ell-2}\Bigg\langle \imath(t,x,\phi,\varphi), \frac{ru(t,x,\phi)}{1+kv(t,x,\phi)} - \frac{ru(t,x,\varphi)}{1+kv(t,x,\varphi)} \\
&- \frac{\beta u(t,x,\phi)v(t,x,\phi)}{\epsilon+\delta\gamma+u(t,x,\phi)+\rho v(t,x,\phi)} + \frac{\beta u(t,x,\varphi)v(t,x,\varphi)}{\epsilon+\delta\gamma+u(t,x,\varphi)+\rho v(t,x,\varphi)}\Bigg\rangle \\
&- \ell\xi_m\|\imath(t,x,\phi,\varphi)\|^{\ell} - \ell\alpha_1\|\imath(t,x,\phi,\varphi)\|^{\ell}(\|u(t,x,\phi)+u(t,x,\varphi)\|) - \ell(\xi_0-\xi_m)\mathrm{e}^{-\theta_1 t}\|\imath(t,x,\phi,\varphi)\|^{\ell} \\
&+ \ell\|\jmath(t,x,\phi,\varphi)\|^{\ell-2}\Bigg\langle \jmath(t,x,\phi,\varphi), \frac{c\beta(u(t,x,\phi)+\gamma)v(t,x,\phi)}{\epsilon+\delta\gamma+u(t,x,\phi)+\rho v(t,x,\phi)} \\
&- \frac{c\beta(u(t,x,\varphi)+\gamma)v(t,x,\varphi)}{\epsilon+\delta\gamma+u(t,x,\varphi)+\rho v(t,x,\varphi)} - \eta u(t,x,\phi)v(t,x,\phi) + \eta u(t,x,\varphi)v(t,x,\varphi)\Bigg\rangle \\
&- \ell\zeta_m\|\jmath(t,x,\phi,\varphi)\|^{\ell} - \ell\alpha_2\|\jmath(t,x,\phi,\varphi)\|^{\ell}(\|v(t,x,\phi)+v(t,x,\varphi)\|) - \ell(\zeta_0-\zeta_m)\mathrm{e}^{-\theta_2 t}\|\jmath(t,x,\phi,\varphi)\|^{\ell} \\
&+ \ell d_u\|\imath(t,x,\phi,\varphi)\|^{\ell-2}\langle \Delta\imath(t,x,\phi,\varphi), \imath(t,x,\phi,\varphi)\rangle + \frac{1}{2}\ell(\ell-1)\sigma_1^2(t)\|\imath(t,x,\phi,\varphi)\|^{\ell} \\
&+ \ell d_v\|\jmath(t,x,\phi,\varphi)\|^{\ell-2}\langle \Delta\jmath(t,x,\phi,\varphi), \jmath(t,x,\phi,\varphi)\rangle + \frac{1}{2}\ell(\ell-1)\sigma_2^2(t)\|\jmath(t,x,\phi,\varphi)\|^{\ell}\Bigg]\mathrm{d}t \\
&- \ell\sigma_1(t)\|\imath(t,x,\phi,\varphi)\|^{\ell}\mathrm{e}^{\lambda t}\mathrm{d}W_1(t) - \ell\sigma_2(t)\|\jmath(t,x,\phi,\varphi)\|^{\ell}\mathrm{e}^{\lambda t}\mathrm{d}W_2(t) \\
\leq& \lambda\Xi(t,x,\phi,\varphi)\mathrm{d}t + \mathrm{e}^{\lambda t}\Bigg[\ell\|\imath(t,x,\phi,\varphi)\|^{\ell-2}\langle \imath(t,x,\phi,\varphi), P_1\rangle + \ell\|\jmath(t,x,\phi,\varphi)\|^{\ell-2}\langle \jmath(t,x,\phi,\varphi), P_2\rangle \\
&+ \ell\alpha_1\|\imath(t,x,\phi,\varphi)\|^{\ell}(\|u(t,x,\phi)+u(t,x,\varphi)\|) + \ell\alpha_2\|\jmath(t,x,\phi,\varphi)\|^{\ell}(\|v(t,x,\phi)+v(t,x,\varphi)\|) \\
&- \ell d_u\|\imath(t,x,\phi,\varphi)\|^{\ell-2}\|\nabla\imath(t,x,\phi,\varphi)\|^2 - \ell d_v\|\jmath(t,x,\phi,\varphi)\|^{\ell-2}\|\nabla\jmath(t,x,\phi,\varphi)\|^2 \\
&- \ell\xi_m\|\imath(t,x,\phi,\varphi)\|^{\ell} - \ell(\xi_0-\xi_m)\mathrm{e}^{-\theta_1 t}\|\imath(t,x,\phi,\varphi)\|^{\ell} - \ell\zeta_m\|\jmath(t,x,\phi,\varphi)\|^{\ell} - \ell(\zeta_0-\zeta_m)\mathrm{e}^{-\theta_2 t}\|\jmath(t,x,\phi,\varphi)\|^{\ell} \\
&+ \frac{1}{2}\ell(\ell-1)\sigma_1^2(t)\|\imath(t,x,\phi,\varphi)\|^{\ell} + \frac{1}{2}\ell(\ell-1)\sigma_2^2(t)\|\jmath(t,x,\phi,\varphi)\|^{\ell}\Bigg]\mathrm{d}t \\
&- \ell\sigma_1(t)\|\imath(t,x,\phi,\varphi)\|^{\ell}\mathrm{e}^{\lambda t}\mathrm{d}W_1(t) - \ell\sigma_2(t)\|\jmath(t,x,\phi,\varphi)\|^{\ell}\mathrm{e}^{\lambda t}\mathrm{d}W_2(t),
\end{aligned}
\tag{5}
$$

where

$$P_1 = \frac{ru(t,x,\phi)}{1+kv(t,x,\phi)} - \frac{ru(t,x,\varphi)}{1+kv(t,x,\varphi)} - \frac{\beta u(t,x,\phi)v(t,x,\phi)}{\epsilon+\delta\gamma+u(t,x,\phi)+\rho v(t,x,\phi)} + \frac{\beta u(t,x,\varphi)v(t,x,\varphi)}{\epsilon+\delta\gamma+u(t,x,\varphi)+\rho v(t,x,\varphi)},$$

$$P_2 = \frac{c\beta(u(t,x,\phi)+\gamma)v(t,x,\phi)}{\epsilon+\delta\gamma+u(t,x,\phi)+\rho v(t,x,\phi)} - \frac{c\beta(u(t,x,\varphi)+\gamma)v(t,x,\varphi)}{\epsilon+\delta\gamma+u(t,x,\varphi)+\rho v(t,x,\varphi)} - \eta u(t,x,\phi)v(t,x,\phi) + \eta u(t,x,\varphi)v(t,x,\varphi).$$

Note that

$$
\begin{aligned}
&\ell\|\imath(t,x,\phi,\varphi)\|^{\ell-2}\Big\langle \imath(t,x,\phi,\varphi), P_1\Big\rangle \\
\leq& \ell\|\imath(t,x,\phi,\varphi)\|^{\ell-2}\Bigg\langle \imath(t,x,\phi,\varphi), \frac{r(1+kv(t,x,\phi))\imath(t,x,\phi,\varphi)}{(1+kv(t,x,\phi))(1+kv(t,x,\varphi))} + \frac{rku(t,x,\phi)\jmath(t,x,\phi,\varphi)}{(1+kv(t,x,\phi))(1+kv(t,x,\varphi))} \\
&+ \frac{\beta[\epsilon+\delta\gamma+\rho v(t,x,\phi)]v(t,x,\varphi)\imath(t,x,\phi,\varphi) + \beta[\epsilon+\delta\gamma+u(t,x,\varphi)]u(t,x,\phi)\jmath(t,x,\phi,\varphi)}{(\epsilon+\delta\gamma+u(t,x,\phi)+\rho v(t,x,\phi))(\epsilon+\delta\gamma+u(t,x,\varphi)+\rho v(t,x,\varphi))}\Bigg\rangle \\
\leq& \ell\left(r+\frac{\beta}{\rho}\right)\|\imath(t,x,\phi,\varphi)\|^{\ell} + \ell(rkK+\beta)\|\imath(t,x,\phi,\varphi)\|^{\ell-1}\|\jmath(t,x,\phi,\varphi)\|.
\end{aligned}
$$

Then, using Young inequality leads to

$$\ell(rkK+\beta)\|\imath(t,x,\phi,\varphi)\|^{\ell-1}\|\jmath(t,x,\phi,\varphi)\| \leq (\ell-1)(rkK+\beta)\|\imath(t,x,\phi,\varphi)\|^{\ell} + (rkK+\beta)\|\jmath(t,x,\phi,\varphi)\|^{\ell}.$$

Thus

$$\ell\|\imath(t,x,\phi,\varphi)\|^{\ell-2}\Big\langle \imath(t,x,\phi,\varphi), P_1\Big\rangle \leq \left[\ell\left(r+\frac{\beta}{\rho}\right) + (\ell-1)(rkK+\beta)\right]\|\imath(t,x,\phi,\varphi)\|^{\ell} + (rkK+\beta)\|\jmath(t,x,\phi,\varphi)\|^{\ell}. \tag{6}$$

Similarly

$$
\begin{aligned}
&\ell\|\jmath(t,x,\phi,\varphi)\|^{\ell-2}\Big\langle \jmath(t,x,\phi,\varphi), P_2\Big\rangle\\
\le&\ell\|\jmath(t,x,\phi,\varphi)\|^{\ell-2}\Bigg\langle \jmath(t,x,\phi,\varphi), \frac{c\beta\gamma\left[(\epsilon+\delta\gamma+u(t,x,\varphi))\jmath(t,x,\phi,\varphi)+v(t,x,\varphi)\imath(t,x,\phi,\varphi)\right]}{(\epsilon+\delta\gamma+u(t,x,\phi)+\rho v(t,x,\phi))(\epsilon+\delta\gamma+u(t,x,\varphi)+\rho v(t,x,\varphi))}\\
&+\frac{c\beta[\epsilon+\delta\gamma+\rho v(t,x,\varphi)]v(t,x,\phi)\imath(t,x,\phi,\varphi)+c\beta[\epsilon+\delta\gamma+u(t,x,\phi)]u(t,x,\varphi)\jmath(t,x,\phi,\varphi)}{(\epsilon+\delta\gamma+u(t,x,\phi)+\rho v(t,x,\phi))(\epsilon+\delta\gamma+u(t,x,\varphi)+\rho v(t,x,\varphi))}\\
&+\eta u(t,x,\phi)\jmath(t,x,\phi,\varphi)+\eta v(t,x,\varphi)\imath(t,x,\phi,\varphi)\Bigg\rangle\\
\le&\left(\frac{\ell c\beta}{\rho}+\frac{c\beta\gamma}{\rho(\epsilon+\delta\gamma)}+\eta K\right)\|\imath(t,x,\phi,\varphi)\|^{\ell}+\left(\frac{c\beta\gamma[\ell(1+\rho)-1]}{\rho(\epsilon+\delta\gamma)}+\eta K\right)\|\jmath(t,x,\phi,\varphi)\|^{\ell}.
\end{aligned}
\tag{7}
$$

Substituting (6) and (7) into (5), and then integrating both sides and taking expectations, one gets

$$
\mathbb{E}\Xi(t,x,\phi,\varphi)\le\mathbb{E}\Xi(0,x,\phi,\varphi)+\lambda\mathbb{E}\int_0^t\Theta(r,x,\phi,\varphi)\mathrm{d}r+H_1\mathbb{E}\int_0^t \mathrm{e}^{\lambda r}\|\imath(r,x,\phi,\varphi)\|^{\ell}\mathrm{d}r+H_2\mathbb{E}\int_0^t \mathrm{e}^{\lambda r}\|\jmath(r,x,\phi,\varphi)\|^{\ell}\mathrm{d}r. \tag{8}
$$

Taking the $\sup_{\phi,\varphi\in\mathbb{U}}$ on (8) results in

$$
\sup_{\phi,\varphi\in\mathbb{U}}\mathbb{E}\Xi(t,x,\phi,\varphi)\le\sup_{\phi,\varphi\in\mathbb{U}}\mathbb{E}\Xi(0,x,\phi,\varphi)+(\lambda+C_3)\sup_{\phi,\varphi\in\mathbb{U}}\mathbb{E}\int_0^t\Xi(r,x,\phi,\varphi)\mathrm{d}r,
$$

where $C_3=\max\{H_1,H_2\}$.

Using the Gronwall's inequality leads to

$$
\sup_{\phi,\varphi\in\mathbb{U}}\mathbb{E}\Xi(t,x,\phi,\varphi)\le\sup_{\phi,\varphi\in\mathbb{U}}\mathbb{E}\Xi(0,x,\phi,\varphi)\mathrm{e}^{(\lambda+C_3)t}.
$$

That is

$$
\sup_{\phi,\varphi\in\mathbb{U}}\mathbb{E}\left(\|\imath(t,x,\phi,\varphi)\|^{\ell}+\|\jmath(t,x,\phi,\varphi)\|^{\ell}\right)\le\sup_{\phi,\varphi\in\mathbb{U}}\mathbb{E}\Xi(0,x,\phi,\varphi)\mathrm{e}^{C_4t}.
$$

Assumption $C_4<0$, then when $t\to\infty$, one has

$$
\lim_{t\to\infty}\sup_{\phi,\varphi\in\mathbb{U}}\mathbb{E}\left(\|\imath(t,x,\phi,\varphi)\|^{\ell}+\|\jmath(t,x,\phi,\varphi)\|^{\ell}\right)=0.
$$

Therefore, there has a stationary distribution $\pi\in\mathcal{P}(\mathbb{H})$.

Now, we assume that $\pi'\in\mathcal{P}(\mathbb{H})$ is another stationary distribution of model (3). Then, one has

$$
|\pi(g)-\pi'(g)|\le\int_{\mathbb{H}\times\mathbb{H}}|\mathbb{P}_tg(\phi)-\mathbb{P}_tg(\varphi)|\pi(\mathrm{d}\phi)\pi'(\mathrm{d}\varphi)\le C_5\mathrm{e}^{-\lambda t},
$$

where $C_5>0$.

Thus

$$
\lim_{t\to\infty}|\pi(g)-\pi'(g)|=0.
$$

Therefore, we can deduce that the model (3) has a unique stationary distribution. □

3. Conclusion

This paper investigates the conditions for the existence and uniqueness of stationary distribution of a stochastic reaction–diffusion predator–prey model with additional food, fear effect, anti-predator behavior and Ornstein–Uhlenbeck process according to a suitable Lyapunov functionals. This provides theoretical guidance for exploring the long-term behavior and stability of diffusion population dynamics models under the interference of Ornstein–Uhlenbeck process. Moreover, this study extends the work of Qi and Liu [7], innovatively integrates the anti-predator behavior of the population, and uses the more general Ornstein–Uhlenbeck process instead of the more ideal white noise to accurately capture the behavior pattern of the population in the natural environment.

Data availability

No data was used for the research described in the article.

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